

Modular dynamical systems on networks

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Abstract

We propose a new framework for the study of continuous time dynamical systems on networks. We view such dynamical systems as collections of interacting control systems. We show that a class of maps between graphs called **graph fibrations** give rise to maps between dynamical systems on networks. This allows us to produce conjugacy between dynamical systems out of combinatorial data. In particular we show that surjective graph fibrations lead to synchrony subspaces in networks. The injective graph fibrations, on the other hand, give rise to surjective maps from large dynamical systems to smaller ones. One can view these surjections as a kind of “fast/slow” variable decompositions or as “abstractions” in the computer science sense of the word.

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1 Introduction

1.1 Overview

A fundamental question in the study of dynamical systems is to determine the existence and properties of a map that intertwines the dynamics of two different systems. Stated concretely, given two manifolds M, N , and two flows $\varphi_t: M \rightarrow M, \psi_t: N \rightarrow N$, does there exist a map $h: M \rightarrow N$ such that

$$h \circ \varphi_t = \psi_t \circ h? \quad (1.1.1)$$

Equivalently, given two manifolds and two vector fields $X: M \rightarrow TM, Y: N \rightarrow TN$, does there exist a map h such that

$$dh \circ X = Y \circ h? \quad (1.1.2)$$

If there is an h that satisfies (1.1.1) or (1.1.2), then we call this a *map of dynamical systems*. Given such a map h , we would like to understand its properties and to compute it explicitly.

A common restriction requires that h be invertible. In this case, it is said that we have exhibited a *conjugacy* between the two dynamical systems, and this means that all of the dynamical features of the flow are the same¹. The notion of conjugacy of dynamical systems goes back at least to Poincaré [4–6], it was further developed by Smale and collaborators [1, 7], and is now the basic notion in modern dynamical systems theory.

A more general notion of conjugacy arises from the relaxation of the assumption of invertibility; here, the existence of the map h still produces significant information. For example, the flow ψ_t has a fixed point iff we can exhibit a map $h: \{*\} \rightarrow N$, where $\{*\}$ is a one-point set, and h satisfies (1.1.1). If $M = S^1$ and the flow φ_t is given by $\varphi_t(e^{i\theta}) = e^{2\pi it/T} e^{i\theta}$, then the existence of $h: M \rightarrow N$ satisfying (1.1.1) amounts to the flow ψ_t having a periodic orbit of period T .

It is also common for h to be chosen to be surjective. In this case the map is typically termed a *semiconjugacy* [2, 8] and certain nice properties follow. We do not expand on this here, but we will exploit the existence of semi-conjugacies for networked systems below in Section 5.

The question of determining whether a map relating two dynamical systems exists, and what its properties might be, is exceedingly difficult [9–12] in general. In some cases, even if such an h is known to exist, determining its form (or even qualitative properties) can be challenging.

In a different direction, dynamical systems defined on networks have become the fundamental object of study across a variety of fields. Some examples include the design of communications networks [13]; cognitive science, computational neuroscience, and robotics (see, for example [14–

¹Note that what we mean by “the same” depends on the category in which we work. For instance if h is a homeomorphism then we say that the dynamical systems are topologically the same. Many of the implications of the existence of a conjugacy are worked out in [1–3].

19]); gene regulatory networks [20–22] and more general complex biochemical networks [23]; and finally in complex active media [24–28].

There are multiple definitions in the literature of what it means to define a “dynamical system on a network” and we will not compare them here. A common thread running through these definitions is that imposing a network structure on a dynamical system should mean that component j of the system depends upon component i of the system iff the underlying graph has an edge $i \rightarrow j$.

In this paper we show that an imposition of a “network structure” on a dynamical system allows us to produce maps between dynamical systems in a precise, computable and combinatorial manner from finite data. Thus the purpose of this manuscript is twofold: first, to present a notion of a dynamical system “consistent with a graph”; second, to show that certain maps between graphs induce maps between the dynamical systems that live on them. In particular, we will show below that all graph maps that respect a particular combinatorial structure induce maps between the dynamical systems living on these graphs.

We focus on the case where the dynamics is modeled by vector fields on manifolds. The interactions of subsystems are coded by directed (multi-)graphs with “labels”. These labels in particular, assign to each node of a graph the phase space of the relevant subsystem. The ideas of the paper can be extended to both discrete-time, hybrid and stochastic systems, and we plan to do so in future work.

As stated above, the main result of the paper is the construction of maps of dynamical systems from maps of labeled graphs. In particular we show that surjective maps of graphs, such as the one arising from quotienting a graph by an appropriate equivalence relation, give rise to embeddings of dynamical systems; second, injective maps of graphs give rise submersions of the corresponding phase spaces and surjective maps of dynamical systems. The former is very useful in characterizing the “modularity” of a networked dynamical system; the latter gives a precise mathematical formulation of some intuitive notions of whether and how we can think of a large dynamical system driven by a subsystem.

1.2 Background and previous work

The present paper is inspired by several distinct bodies of work that are well known in the applied mathematics communities.

The first body of work has been mainly applied to chemical reaction systems, and, in some specific cases, to Petri nets; this work has been used in both the deterministic and stochastic settings. One of the earliest results in this direction is the “zero deficiency theorem” of Feinberg [29–34], first used to show the existence of stable equilibria in biochemical systems and then expanded to statements about the existence and structure of the equilibria in stochastic biochemical systems [35]. This type of methodology has also been expanded to Petri nets [36–38] and models describing gene regulatory networks [21].

The second body of work is due to Golubitsky, Stewart, and various collaborators [39–64]. These authors considered a notion of ODEs (vector fields defined on Euclidean spaces) that were consistent with a graph structure. The resulting networks are called coupled cell systems and the approach the groupoid formalism. The main idea was to consider “balanced” equivalence relations on the vertices of a graph. They showed that these relations lead to the existence of certain invariant subspaces termed “polydiagonals.” We will see that the quotient maps resulting from balanced equivalence relations are instances of graph fibration in the sense of Boldi and Vigna [65]. However, while we are greatly indebted to this body of work for its intellectual inspiration, we also point out that

our approach differs from this work in some very specific ways. We elucidate the connections and contrasts in Remark 4.3.3 below.

An alternative approach to coupled cell systems has been developed by Field and collaborators [66–68]. They also considered ODEs and other types of dynamical systems consistent with directed graphs. This approach is considered broadly equivalent to that of Golubitsky *et al.*

1.3 The contributions of the paper

In this paper we propose a new framework for the study of continuous time dynamical systems on networks. We view such dynamical systems as collections of interacting control systems. We show that a class of maps between graphs called **graph fibrations** give rise to maps between dynamical systems on networks. This allows us to produce conjugacy between dynamical systems out of combinatorial data. While the current work is certainly inspired by the methods and results of both the Feinberg et al., Golubitsky et al. and, to a lesser extent, Field et al. groups, our approach and results differ in several important respects.

1. Our basic philosophy is that of category theory — so rather than study dynamical systems one at a time we aim to study maps between all relevant dynamical systems at once. To quote Silverman [69]:

A meta-mathematical principle is that one first studies (isomorphism classes of) objects, then one studies the maps between objects that preserve the objects’ properties, then the maps themselves become objects for study and one tries to put a “nice” structure on the collection of maps (often modulo some equivalence relation).

2. Our set-up is coordinate-free and works for vector fields on manifolds, not just \mathbb{R}^n . In this case we are enlarging both the scope of the Feinberg et al. work (polynomial vector fields on positive orthants) and that of Golubitsky et al. (vector fields on Euclidean spaces). This aspect of our approach is similar to the work of Field et al. There are several motivations for working on manifolds as opposed to ODEs living in Euclidean spaces. These include dealing effectively with constraints, and extending the results to the setting of geometrical mechanics.
3. It will be evident from the construction below that the quotient maps of graphs by balanced equivalence relations of [61] are special cases of graph fibrations — they are the surjective graph fibrations. However, even in the case of surjective graph fibrations our maps of dynamical systems have the opposite direction from the maps in the groupoid formalism. Rather than restricting from polydiagonals we extend from polydiagonals. This allows us to deal with surjective and general graph fibration on the same footing.

1.4 Motivating example

Consider an ODE in $(\mathbb{R}^n)^3$ of the form

$$\dot{x}_1 = f(x_2), \quad \dot{x}_2 = f(x_1), \quad \dot{x}_3 = f(x_2) \tag{1.4.1}$$

for some smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. That is, consider the flow of the vector field

$$F : (\mathbb{R}^n)^3 \rightarrow (\mathbb{R}^n)^3, \quad F(x_1, x_2, x_3) = (f(x_2), f(x_1), f(x_2)).$$

It is easy to check that F is tangent to the diagonal

$$\mathbb{R}^n \simeq \Delta = \{(x_1, x_2, x_3) \in (\mathbb{R}^n)^3 \mid x_1 = x_2 = x_3\}$$

and that the restriction of the flow of F to Δ is the flow of the ODE

$$\dot{u} = f(u).$$

One can also see another invariant submanifold of F :

$$(\mathbb{R}^n)^2 \simeq \Delta' = \{(x_1, x_2, x_3) \in (\mathbb{R}^n)^3 \mid x_1 = x_3\}.$$

On Δ' the flow of F is the flow of the ODE

$$\dot{v}_1 = f(v_2), \quad \dot{v}_2 = f(v_1).$$

Moreover the projection

$$\pi : (\mathbb{R}^n)^3 \rightarrow \Delta', \quad \pi(x_1, x_2, x_3) = (x_1, x_2, x_1)$$

intertwines the flows of F on $(\mathbb{R}^n)^3$ and on Δ' . We have thus observed two subsystems of $((\mathbb{R}^n)^3, F)$ and three maps between the three dynamical systems:

$$(\Delta, F|_{\Delta}) \hookrightarrow ((\mathbb{R}^n)^3, F) \xrightleftharpoons[\pi]{} (\Delta', F|_{\Delta'}) \quad (1.4.2)$$

Where do these subsystems and maps come from? There is no obvious symmetry of $(\mathbb{R}^n)^3$ that preserves the vector field F and fixes the diagonal Δ and thus could account for the existence of this invariant submanifold. Nor is there any F -preserving symmetry that fixes Δ' . In fact the vector field F does not seem to have any symmetry. The graph G recording the interdependence of the variables (x_1, x_2, x_3) in the ODE (1.4.1) has three vertices and three arrows:

$$G = \begin{array}{c} \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \\ \text{---} \quad \text{---} \quad \text{---} \end{array} \quad (1.4.3)$$

The graph has no non-trivial symmetries. Nonetheless, the existence of the subsystems $(\Delta, F|_{\Delta})$, $(\Delta', F|_{\Delta'})$ and the whole diagram of the dynamical systems (1.4.2) can be deduced from certain properties of the graph G . There are two surjective maps of graphs:

$$\varphi : G \rightarrow \textcircled{}$$

and

$$\psi : G \rightarrow \begin{array}{c} \textcircled{a} \quad \textcircled{b} \\ \text{---} \quad \text{---} \end{array},$$

with ψ defined on the vertices by $\psi(2) = b$, $\psi(1) = a = \psi(3)$, and one embedding

$$\tau : \begin{array}{c} \textcircled{a} \quad \textcircled{b} \\ \text{---} \quad \text{---} \end{array} \hookrightarrow \begin{array}{c} \textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \\ \text{---} \quad \text{---} \quad \text{---} \end{array}.$$

We can collect all of these maps into one diagram

$$\begin{array}{c}
 \text{graph with one node and a self-loop} \xleftarrow{\varphi} \text{graph with nodes 1, 2, 3} \xrightleftharpoons[\psi]{\tau} \text{graph with nodes a, b}
 \end{array} \tag{1.4.4}$$

A comparison of (1.4.2) and (1.4.4) evokes a pattern: for every map which intertwines dynamical systems in (1.4.2), there is a corresponding map of graphs in (1.4.4) with the arrows reversed, and vice versa.

The same pattern holds when we replace the vector space \mathbb{R}^n by an arbitrary manifold M . Given a pair of manifolds U and N , we think of a map $X : U \times N \rightarrow TN$ with $X(u, n) \in T_n N$ as a control system with the points of U controlling the the dynamics on N . Now consider a vector field

$$F : M^3 \rightarrow T(M^3) = TM \times TM \times TM$$

of the form

$$F(x_1, x_2, x_3) = (f(x_2, x_1), f(x_1, x_2), f(x_2, x_3))$$

for some control system

$$f : M \times M \rightarrow TM, \quad \text{with} \quad f(u, v) \in T_v M.$$

Then once again the three maps of graphs in the diagram (1.4.4) give rise to maps of dynamical systems

$$(\Delta_M, F|_{\Delta_M}) \xrightarrow{\quad} (M^3, F) \xrightleftharpoons[\pi]{\quad} (\Delta'_M, F|_{\Delta'_M}) \tag{1.4.5}$$

What accounts for the patterns we have seen? Notice that the dynamical systems (1.4.5) are constructed out of *one* control system $f : M \times M \rightarrow TM$. At the same time, in each of the graphs in (1.4.4), every vertex has exactly *one* incoming arc. This is not a coincidence. The rough idea for the technology which generalizes this example is this: if we have a dynamical system made up of repeated control system modules whose couplings are encoded in graphs, then the appropriate maps of graphs lift to maps of dynamical systems. Making this precise requires a number of constructions and theorems; these make up the bulk of this paper.

1.5 Main ideas of the paper

We study *dependency* and *modularity* of networks and their effect on the concomitant dynamical systems.

By *dependency* we mean the following. Each node in a network corresponds to a single dynamical variable living on a particular manifold. We then require that the variable corresponding to node i can depend on the variable corresponding to node j iff there is an edge in the graph from node j to node i . We give a more precise description of this requirement below and we will denote the space all vector fields with this property by $\mathbb{S}(G, \mathcal{P})$; see Section 2.4 below for a definition.

The rough idea of *modularity* is that if we ever have multiple nodes of the graph that are “the same” and have “the same” inputs, then we require that these nodes are interchangeable in the dynamical system. Speaking more precisely, we will assume that in each network, each node has a “type” (it will, in fact, be a manifold attached to this node which corresponds to the phase space

of the variables associated to that node), and if we ever see two nodes, each with type x , with n inputs in the graph, such that these inputs are of type x_1, \dots, x_n , then the vector field defined on these two nodes must depend on their inputs in exactly the same manner. We will denote all vector fields that respect this principle of modularity as $\mathbb{V}(G, \mathcal{P})$ and give a precise definition of these vector fields in Section 3.

In the example in the previous subsection, the principle of *dependency* tells us that the system living on the graph in (1.4.3) must be of the form

$$\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = g(x_2, x_1), \quad \dot{x}_3 = h(x_3, x_2), \quad (1.5.1)$$

but x_1 cannot depend on x_3 , for example. The principle of *modularity* tells us that the functions f, g, h must all be the same, i.e. that

$$\dot{x}_1 = f(x_1, x_2), \quad \dot{x}_2 = f(x_2, x_1), \quad \dot{x}_3 = f(x_3, x_2), \quad (1.5.2)$$

Therefore, all systems in $\mathbb{S}(G, \mathcal{P})$ must satisfy (1.5.1), but those in $\mathbb{V}(G, \mathcal{P})$ must satisfy the stricter requirements (1.5.2).

2 Networks and dynamics on networks

The goal of this section is to define networks and dynamics on networks in the context of continuous time dynamical systems. It is not uncommon to read that “a network is a graph.” This could not be a complete story since by a network one usually means a collection of interconnected subsystems. We now make this precise.

2.1 Graphs, manifolds and networks

Throughout the paper *graphs* are directed multigraphs, possibly with loops and multiple edges between nodes. More precisely, we use the following definition:

2.1.1 Definition. A graph G consists of two sets G_1 (of arrows, or edges), G_0 (of nodes, or vertices) and two maps $s, t : G_1 \rightarrow G_0$ (source, target):

$$G = \{s, t : G_1 \rightarrow G_0\}.$$

We write $G = \{G_1 \rightrightarrows G_0\}$. A graph G is *finite* if it has finitely many arrows and edges.

2.1.2 Definition. A map of graphs $\varphi : A \rightarrow B$ from a graph A to a graph B is a pair of maps $\varphi_1 : A_1 \rightarrow B_1$, $\varphi_0 : A_0 \rightarrow B_0$ taking edges of A to edges of B , nodes of A to nodes of B so that for any edge γ of A we have

$$\varphi_0(s(\gamma)) = s(\varphi_1(\gamma)) \quad \text{and} \quad \varphi_0(t(\gamma)) = t(\varphi_1(\gamma)).$$

We will usually omit the indices 0 and 1 and write $\varphi(\gamma)$ for $\varphi_1(\gamma)$ and $\varphi(a)$ for $\varphi_0(a)$.

2.1.3 Remark. The collection of graphs and maps of graphs form a category **Graph**. The subcollection of finite graphs and maps of graphs forms a full subcategory **FinGraph**.

To construct a network from a graph we need to attach phase spaces to its vertices. Since we are interested in continuous time dynamical systems, we choose phase spaces to be (finite dimensional paracompact Hausdorff) manifolds. Other choices, of course, may also be reasonable, such as coordinate vector spaces \mathbb{R}^n or manifolds with corners.

2.1.4 Definition (Network). A network is a pair (G, \mathcal{P}) where G is a finite graph and \mathcal{P} is a function that assigns to each node $a \in G_0$ of G a manifold $\mathcal{P}(a)$. We refer to \mathcal{P} as a **phase space function**. Note that the target of the function \mathcal{P} is the collection **Man** of all manifolds: $\mathcal{P} : G_0 \rightarrow \mathbf{Man}$.

A map of networks from (G, \mathcal{P}) to (G', \mathcal{P}') is a map of graphs $\varphi : G \rightarrow G'$ so that

$$\mathcal{P}' \circ \varphi = \mathcal{P}.$$

2.1.5 Remark. It is easy to see that composition of two maps of networks is again a map of networks. In other words networks form a category. We denote it by **FinGraph/Man**.

Given a network (G, \mathcal{P}) as defined above, a state of the network is completely determined by the states of its nodes. Hence the total phase space of the network should be the product

$$\mathbb{P}(G, \mathcal{P}) := \prod_{a \in G_0} \mathcal{P}(a).$$

Note, however, a small issue: there is no natural ordering on the vertices of the graph G . We could *choose* an ordering (a_1, \dots, a_n) of the vertices and define the total phase space as the Cartesian product

$$\mathbb{P}(G, \mathcal{P}) := \prod_{i=1}^n \mathcal{P}(a_i).$$

However, it will be convenient *not* to choose an ordering of vertices and use a slightly different notion of product. This version of the product is used, for example, in chemical reaction network literature [37].

2.1.6 Definition. Given a family $\{M_s\}_{s \in S}$ of manifolds indexed by a finite set S , denote by $\bigsqcup_{s \in S} M_s$ their disjoint union². The **categorical product** of a finite family $\{M_s\}_{s \in S}$ of manifolds is the manifold

$$\prod_{s \in S} M_s := \{x : S \rightarrow \bigsqcup_{s \in S} M_s \mid x(s) \in M_s \text{ for all } s \in S\}.$$

We note that for each index $s \in S$ we have projection maps $\pi_s : \prod_{s' \in S} M_{s'} \rightarrow M_s$ are defined by

$$\pi_s(x) = x(s).$$

These projections are surjective submersions.

We denote $x(s) \in M_s$ by x_s and think of it as s^{th} “coordinate” of an element $x \in \prod_{s \in S} M_s$. Equivalently we may think of elements of the categorical product $\prod_{s \in S} M_s$ as **unordered** tuples $(x_s)_{s \in S}$ with $x_s \in M_s$.

² The disjoint union may be defined by $\bigsqcup_{s \in S} M_s := \bigcup_{s \in S} (M_s \times \{s\})$.

2.1.7 Remark. It is not hard to show that the categorical product, as defined above, has the following universal property: given a manifold N and a family of smooth maps $\{f_s : N \rightarrow M_s\}_{s \in S}$ there is a unique map $f : N \rightarrow \prod_{s \in S} M_s$ with

$$\pi_s \circ f = f_s \quad \text{for all } s \in S.$$

In fact categorical products are usually *defined* by this universal property [70].

2.1.8 Remark. For a family $\{M_s\}_{s \in S}$ of manifolds indexed by a finite set S every ordering $\{s_1, \dots, s_n\}$ of elements of S identifies the categorical product $\prod_{s \in S} M_s$ (as a manifold) with the Cartesian product $M_{s_1} \times \dots \times M_{s_n}$.

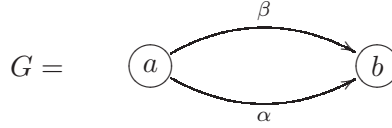
We are now in position to state:

2.1.9 Definition (total phase space of a network (G, \mathcal{P})). For a pair (G, \mathcal{P}) consisting of a graph G and a phase space function \mathcal{P} we define the **total phase space** of the network (G, \mathcal{P}) to be the manifold

$$\mathbb{P}G \equiv \mathbb{P}(G, \mathcal{P}) := \prod_{a \in G_0} \mathcal{P}(a),$$

the categorical product of manifolds attached to the nodes of the graph G by the phase space function \mathcal{P} .

2.1.10 Example. Consider the graph



Define $\mathcal{P} : G_0 \rightarrow \mathbf{Man}$ by $\mathcal{P}(a) = S^2$ (the two sphere) and $\mathcal{P}(b) = S^3$. Then the total phase space $\mathbb{P}(G, \mathcal{P})$ is the Cartesian product $S^2 \times S^3$.

2.1.11 Notation. If $G = \{\emptyset \rightrightarrows \{a\}\}$ is a graph with one node a and no arrows, we write $G = \{a\}$. Then for any phase space function $\mathcal{P} : G_0 = \{a\} \rightarrow \mathbf{Man}$ we abbreviate $\mathbb{P}(\{\emptyset \rightrightarrows \{a\}\}, \mathcal{P} : \{a\} \rightarrow \mathbf{Man}) = \mathbb{P}(\{a\}, \mathcal{P} : \{a\} \rightarrow \mathbf{Man})$ as $\mathbb{P}a$.

2.1.12 Proposition. A map of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ naturally defines a map of corresponding total phase spaces

$$\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G.$$

Proof. We use the universal property of the product $\mathbb{P}G = \prod_{a \in G_0} \mathcal{P}(a)$: to define a map f from any manifold N to $\mathbb{P}G$ it is enough to define a family of maps $\{f_a : N \rightarrow \mathcal{P}(a)\}_{a \in G_0}$. For any node a' of G' we have the canonical projection

$$\pi'_{a'} : \mathbb{P}G' \rightarrow \mathcal{P}'(a').$$

We therefore define

$$(\mathbb{P}\varphi)_a := \pi'_{\varphi(a)} : \mathbb{P}G' \rightarrow \mathcal{P}'(\varphi(a)) = \mathcal{P}(a)$$

for all $a \in G_0$. □

2.1.13 Example. Suppose G is a graph with two nodes a, b and no edges, G' is a graph with one node $\{c\}$ and no edges, $\mathcal{P}'(c)$ is a manifold M , $\varphi : G \rightarrow G'$ is the only possible map of graphs (it sends both nodes to c), and $\mathcal{P} : G_0 \rightarrow \mathbf{Man}$ is given by $\mathcal{P}(a) = M = \mathcal{P}(b)$ (so that $\mathcal{P}' \circ \varphi = \mathcal{P}$). Then $\mathbb{P}G' \simeq M$,

$$\mathbb{P}G = \{(x_a, x_b) \mid x_a \in \mathcal{P}(a), x_b \in \mathcal{P}(b)\} \simeq M \times M$$

and $\mathbb{P}\varphi : M \rightarrow M \times M$ is the unique map with $(\mathbb{P}\varphi(x))_a = x$ and $(\mathbb{P}\varphi(x))_b = x$ for all $x \in \mathbb{P}G'$. Thus $\mathbb{P}\varphi : M \rightarrow M \times M$ is the diagonal map $x \mapsto (x, x)$.

2.1.14 Example. Let $(G, \mathcal{P}), (G', \mathcal{P}')$ be as in Example 2.1.13 above and $\psi : (G', \mathcal{P}') \rightarrow (G, \mathcal{P})$ be the map that sends the node c to a . Then $\mathbb{P}\psi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is the map that sends (x_a, x_b) to x_a .

2.1.15 Remark. It is not hard to show that if a map of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is surjective on vertices then $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is an **embedding**. If, on the other hand, φ is injective on vertices, then $\mathbb{P}\varphi$ is a **surjective submersion**.

2.1.16 Remark. The total phase space map $\mathbb{P} : \mathbf{FinGraph}/\mathbf{Man} \rightarrow \mathbf{Man}$ is a contravariant functor: given two maps of networks

$$(G, \mathcal{P}) \xrightarrow{\varphi} (G', \mathcal{P}') \xrightarrow{\psi} (G'', \mathcal{P}'')$$

we have

$$\mathbb{P}(\psi \circ \varphi) = \mathbb{P}\varphi \circ \mathbb{P}\psi. \quad (2.1.1)$$

To indicate that \mathbb{P} reverses the direction of maps we now and subsequently write

$$\mathbb{P} : (\mathbf{FinGraph}/\mathbf{Man})^{\text{op}} \rightarrow \mathbf{Man}.$$

The superscript $^{\text{op}}$ stands for the opposite category, i.e., the category with the same objects but all the arrows reversed.

2.2 Open systems and their interconnections

Having set up a consistent way of assigning phase spaces to graphs (that is, having set up networks of manifolds) we now take up a construction of continuous time dynamical systems compatible with the structure of the network. We build vector fields on total phase spaces of networks by interconnecting appropriate open systems. Our notion of interconnection is borrowed, to some extent, from the control theory literature. See, for example, Willems [71]. We therefore start by recalling a definition of an open (control) systems, which is essentially due to Brockett [72].

2.2.1 Definition. A continuous time control system (or an open system) on a manifold M is a surjective submersion $p : Q \rightarrow M$ and a smooth map $F : Q \rightarrow TM$ so that

$$F(q) \in T_{p(q)}M$$

for all $q \in Q$. (cf., for example, [73]). That is, the following diagram

$$\begin{array}{ccc} Q & \xrightarrow{F} & TM \\ & \searrow p & \downarrow \pi \\ & & M \end{array} \quad \text{commutes,}$$

where $\pi : TM \rightarrow M$ is the canonical projection.

2.2.2. Given a manifold U of “control variables” we may consider control systems of the form

$$F : M \times U \rightarrow TM. \quad (2.2.1)$$

Here the submersion $p : M \times U \rightarrow M$ is given by

$$p(x, u) = x.$$

The collection of all such control systems forms a vector space that we denote by $\text{Ctrl}(M \times U \rightarrow M)$:

$$\text{Ctrl}(M \times U \rightarrow M) := \{F : M \times U \rightarrow TM \mid F(x, u) \in T_x M\}.$$

2.2.3 Notation (Space of sections of a vector bundle). Given a vector bundle $E \rightarrow M$ we denote the space of sections of $E \rightarrow M$ by ΓE or by $\Gamma(E)$.

Now suppose we are given a finite family $\{F_i : M_i \times U_i \rightarrow TM_i\}_{i=1}^N$ of control systems and we want to somehow interconnect them to obtain a closed system $\mathcal{J}(F_1, \dots, F_N)$. This closed system is a vector field on the product $\prod_i M_i$. That is, $\mathcal{J}(F_1, \dots, F_N)$ is a section of the tangent bundle $T(\prod_i M_i) \rightarrow \prod_i M_i$. What additional data do we need to define the interconnection map

$$\mathcal{J} : \prod_i \text{Ctrl}(M_i \times U_i \rightarrow M_i) \rightarrow \Gamma(T(\prod_i M_i)) \quad ?$$

The answer is given by the following proposition:

2.2.4 Proposition. *Given a family $\{p_j : M_j \times U_j \rightarrow M_j\}_{j=1}^N$ of projections and a family of smooth maps $\{s_j : \prod_i M_i \rightarrow M_j \times U_j\}$ so that the diagrams*

$$\begin{array}{ccc} M_j \times U_j & & \\ s_j \uparrow & \searrow p_j & \\ \prod_i M_i & \xrightarrow{pr_j} & M_j \end{array}$$

commute for each index j , there is an interconnection map \mathcal{J} making the diagrams

$$\begin{array}{ccc} \prod_i \text{Ctrl}(M_i \times U_i \rightarrow M_i) & \xrightarrow{\mathcal{J}} & \Gamma(T(\prod_i M_i)) \\ \downarrow \pi_j & & \downarrow \varpi_j = D(pr_j) \circ - \\ \text{Ctrl}(M_j \times U_j \rightarrow M_j) & \xrightarrow{\mathcal{J}_j} & \text{Ctrl}(\prod_i M_i \xrightarrow{pr_j} M_j) \end{array}$$

commute for each j . The components \mathcal{J}_j of the interconnection map \mathcal{J} are defined by $\mathcal{J}_j(F_j) := F_j \circ s_j$ for all j .

Proof. The space of vector fields $\Gamma(T(\prod_i M_i))$ on the product $\prod_i M_i$ is the product of vector spaces $\text{Ctrl}(\prod_i M_i \rightarrow M_j)$:

$$\Gamma(T(\prod_i M_i)) = \prod_j \text{Ctrl}(\prod_i M_i \xrightarrow{pr_j} M_j).$$

In other words a vector field X on the product $\prod_i M_i$ is a tuple $X = (X_1, \dots, X_N)$, where

$$X_j := D(pr_j) \circ X.$$

($D(pr_j) : T\prod M_i \rightarrow TM_j$) denotes the differential of the canonical projection $pr_j : \prod M_i \rightarrow M_j$.) Each component $X_j : \prod_i M_i \rightarrow TM_i$ is a control system.

To define a map from a vector space into a product of vector spaces it is enough to define a map into each of the factors. We have canonical projections

$$\pi_j : \prod_i \text{Ctrl}(M_i \times U_i \rightarrow M_i) \rightarrow \text{Ctrl}(M_j \times U_j \rightarrow M_j), \quad j = 1, \dots, N.$$

Consequently to define the interconnection map \mathcal{J} it is enough to define the maps

$$\mathcal{J}_j : \text{Ctrl}(M_j \times U_j \rightarrow M_j) \rightarrow \text{Ctrl}(\prod_i M_i \xrightarrow{pr_j} M_j).$$

for each index j . We therefore define the maps $\mathcal{J}_j : \text{Ctrl}(M_j \times U_j \rightarrow M_j) \rightarrow \text{Ctrl}(\prod_i M_i \xrightarrow{pr_j} M_j)$, $1 \leq j \leq N$, by

$$\mathcal{J}_j(F_j) := F_j \circ s_j.$$

□

2.2.5 Remark. It will be useful for us to remember that the canonical projections

$$\varpi_j : \Gamma(T\prod M_i) \rightarrow \text{Ctrl}(\prod M_i \rightarrow M_j)$$

are given by

$$\varpi_j(X) = D(pr_j) \circ X,$$

where $D(pr_j) : T\prod M_i \rightarrow TM_j$ are the differential of the canonical projections $pr_j : \prod M_i \rightarrow M_j$.

2.3 Interconnections and graphs

We next explain how networks of manifolds give rise to interconnection maps. To do this precisely it is useful to have a notion of *input trees* of a directed graph. Given a graph, an input tree $I(a)$ of a vertex a is roughly, the vertex itself and all of the arrows leading into it. We want to think of this as a graph in its own right, as follows.

2.3.1 Definition (Input tree). The *input tree* $I(a)$ at a vertex a of a graph G is the graph with the set of vertices $I(a)_0$ given by

$$I(a)_0 := \{a\} \sqcup t^{-1}(a);$$

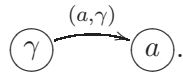
where, as before, the set $t^{-1}(a)$ is the set of arrows in G with target a . The set of edges $I(a)_1$ of the input tree is the set of pairs

$$I(a)_1 := \{(a, \gamma) \mid \gamma \in G_1, \ t(\gamma) = a\},$$

and the source and target maps $I(a)_1 \rightrightarrows I(a)_0$ are defined by

$$s(a, \gamma) = \gamma \quad \text{and} \quad t(a, \gamma) = a.$$

In pictures,



2.3.2 Example. Consider the graph $G = \begin{array}{c} \textcircled{a} \xrightarrow{\beta} \textcircled{b} \\ \textcircled{a} \xrightarrow{\alpha} \textcircled{b} \end{array}$ as in Example 2.1.10. Then the input tree $I(a)$ is the graph with one node a and no edges:

$$I(a) = \textcircled{a}.$$

The input tree $I(b)$ has three nodes and two edges:

$$I(b) = \begin{array}{c} \textcircled{\alpha} \xrightarrow{(b,\alpha)} \textcircled{b} \\ \textcircled{\beta} \xrightarrow{(b,\beta)} \textcircled{b} \end{array}.$$

2.3.3 Remark. For each node a of a graph G we have a natural map of graphs

$$\xi = \xi_a : I(a) \rightarrow G, \quad \xi_a(a, \gamma) = \gamma.$$

We stress that this need not be injective on nodes. For instance in Example 2.3.2 the map $\xi_b : I(b) \rightarrow G$ sends the *nodes* α and β to the same node a .

2.3.4 Example. Consider the graph

$$G = \begin{array}{c} \textcircled{1} \xrightarrow{\alpha} \textcircled{2} \xrightarrow{\gamma} \textcircled{3} \xrightarrow{\epsilon} \textcircled{4} \\ \textcircled{1} \xrightarrow{\beta} \textcircled{2} \xrightarrow{\zeta} \textcircled{4} \\ \textcircled{1} \xrightarrow{\delta} \textcircled{4} \end{array}, \quad (2.3.1)$$

The four input trees of G are the graphs

$$\begin{array}{c} \textcircled{1}, \quad \begin{array}{c} \textcircled{\alpha} \\ \textcircled{\beta} \end{array} \rightarrow \textcircled{2}, \quad \textcircled{\gamma} \rightarrow \textcircled{3}, \quad \begin{array}{c} \textcircled{\epsilon} \\ \textcircled{\zeta} \\ \textcircled{\delta} \end{array} \rightarrow \textcircled{4}. \end{array}$$

2.3.5 Remark. The input tree $I(a)$ of a graph G is a directed tree of height 1: for any vertex x of $I(a)$ with $x \neq a$ there is exactly one edge with source x and target a . Also a is the only vertex of $I(a)$ which is not a source of any edge (it's a *root* of $I(a)$), and all the other vertices of $I(a)$ cannot be targets of any edges (they are *leaves* of $I(a)$).

2.3.6 Remark. It follows from Remark 2.3.5 above that if $\varphi : I(a) \rightarrow I(b)$ is an isomorphism of two input trees (these graphs may be input trees of two different graphs) then necessarily $\varphi(a) = b$

2.3.7 Remark. Given a network (G, \mathcal{P}) and a map of graphs $\varphi : H \rightarrow G$ we get a map of networks

$$\varphi : (H, \mathcal{P} \circ \varphi) \rightarrow (G, \mathcal{P}),$$

hence a map of manifolds

$$\mathbb{P}\varphi : \mathbb{P}G \rightarrow \mathbb{P}H.$$

2.3.8. Let (G, \mathcal{P}) be a network and let a be a node of G . Consider the graph $\{a\}$ with one node and no arrows. Denote the inclusion of $\{a\}$ in G by ι_a and the inclusion into its input tree $I(a)$ by j_a . Then the diagram

$$\begin{array}{ccc} \{a\} & \xrightarrow{j_a} & I(a) \\ & \searrow \iota_a & \swarrow \xi_a \\ & G & \end{array}$$

commutes. By Remarks 2.3.7 and 2.1.16 we have a commuting diagram of maps of manifolds

$$\begin{array}{ccc} \mathbb{P}\{a\} & \xleftarrow{\mathbb{P}j_a} & \mathbb{P}I(a) \\ & \nwarrow \mathbb{P}\iota_a & \nearrow \mathbb{P}\xi_a \\ & \mathbb{P}G & \end{array}$$

2.3.9. Let us now examine more closely the map $\mathbb{P}j_a : \mathbb{P}I(a) \rightarrow \mathbb{P}a$ in 2.3.8 above. Since the set of nodes $I(a)_0$ of the input tree $I(a)$ is the disjoint union

$$I(a)_0 = \{a\} \sqcup t^{-1}(a),$$

and since $\xi_a(\gamma)$ is the source $s(\gamma)$ for any $\gamma \in t^{-1}(a) \subset I(a)_0$, we have

$$\mathbb{P}I(a) = \mathcal{P}(a) \times \prod_{\gamma \in t^{-1}(a)} \mathcal{P}(s(\gamma)).$$

Since $j_a : \{a\} \rightarrow I(a)_0 = \{a\} \sqcup t^{-1}(a)$ is the inclusion,

$$\mathbb{P}j_a : \mathbb{P}I(a) \rightarrow \mathbb{P}a$$

is the projection

$$\mathcal{P}(a) \times \prod_{\gamma \in t^{-1}(a)} \mathcal{P}(s(\gamma)) \rightarrow \mathcal{P}(a).$$

Similarly

$$\mathbb{P}\iota_a : \mathbb{P}G \rightarrow \mathbb{P}a$$

is the projection

$$\prod_{b \in G_0} \mathcal{P}(b) \rightarrow \mathcal{P}(a).$$

Putting 2.3.8 and 2.3.9 together we get

2.3.10 Proposition. *For each node a of a network (G, \mathcal{P}) the diagrams*

$$\begin{array}{ccc} \mathbb{P}I(a) = \mathcal{P}(a) \times \prod_{\gamma \in t^{-1}(a)} \mathcal{P}(s(\gamma)) & \xrightarrow{\mathbb{P}j_a} & \mathcal{P}(a) \\ \uparrow \mathbb{P}\xi_a & \nearrow \mathbb{P}\iota_a & \\ \prod_{b \in G_0} \mathcal{P}(b) & & \end{array}$$

commute.

2.3.11 Example. Suppose $G = \begin{array}{c} \textcircled{a} \quad \textcircled{b} \\ \beta \nearrow \quad \nwarrow \alpha \end{array}$ is a graph as in Example 2.1.10 and suppose $\mathcal{P} : G_0 \rightarrow \text{Man}$ is a phase space function. Then

$$\mathbb{P}I(b) = \mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b),$$

$\mathbb{P}j_b$ is the projection $\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \rightarrow \mathcal{P}(b)$ and

$$\text{Ctrl}(\mathbb{P}I(b) \rightarrow \mathbb{P}b) = \text{Ctrl}(\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \rightarrow \mathcal{P}(b)).$$

On the other hand $\mathbb{P}I(a) = \mathcal{P}(a)$, $\mathbb{P}j_a : \mathcal{P}(a) \rightarrow \mathcal{P}(a)$ is the identity map and

$$\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) = \Gamma(T\mathcal{P}(a)),$$

the space of sections of the tangent bundle $T\mathcal{P}(a)$, that is, the space of vector fields on the manifold $\mathcal{P}(a)$.

2.4 Dependency

Given a network (G, \mathcal{P}) we have a product of vector spaces

$$\prod_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a).$$

The elements of the product are unordered tuples of $(w_a)_{a \in G_0}$ of control systems (cf. 2.1.6). We may think of them as sections of the vector bundle

$$\text{Control}(G, \mathcal{P}) := \bigsqcup_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \rightarrow G_0 \quad (2.4.1)$$

over the vertices of G . This is the main reason for thinking of the collection of infinite dimensional vector spaces $\{\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)\}_{a \in G_0}$ as a vector bundle over a finite set. It will be convenient to have a notation for the space of sections of the bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$.

2.4.1 Definition. Let (G, \mathcal{P}) be a network, as above. We refer to the bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$ as the *control bundle* of the network (G, \mathcal{P}) . We call the sections $(w_a)_{a \in G_0}$ of the control bundle *virtual vector fields* on the network (G, \mathcal{P}) . We denote the space of sections by $\mathbb{S}(G, \mathcal{P})$. Thus

$$\mathbb{S}(G, \mathcal{P}) := \prod_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a).$$

We now argue that an application of the interconnection map $\mathcal{J} : \mathbb{S}(G, \mathcal{P}) \rightarrow \Gamma T(\mathbb{P}(G, \mathcal{P}))$ turns these “virtual vector fields” into actual vector fields on the total phase space $\mathbb{P}(G, \mathcal{P})$ of the network. Indeed observe that Propositions 2.2.4 and 2.3.10 give us

2.4.2 Theorem. *For a network (G, \mathcal{P}) there exists a natural interconnection map*

$$\mathcal{J} : \prod_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \rightarrow \Gamma(T\mathbb{P}G)$$

with

$$\varpi_a \circ \mathcal{J}((w_b)_{b \in G_0}) = w_a \circ \mathbb{P}j_a$$

for all nodes $a \in G_0$. Here $\varpi_a : \Gamma(T\mathbb{P}G) \rightarrow \text{Ctrl}(\mathbb{P}G_0 \xrightarrow{\mathbb{P}\iota_a} \mathbb{P}a)$ are the projection maps defined by $\varpi_a(X) = D(\mathbb{P}\iota_a) \circ X$ (q.v. Remark 2.2.5).

2.4.3 Example. Consider the graph $G = \begin{array}{c} \text{---} \beta \text{---} \\ a \quad \quad b \\ \text{---} \alpha \text{---} \end{array}$ as in Example 2.1.10 with a phase space function $\mathcal{P} : G_0 \rightarrow \text{Man}$. Then the vector field

$$X = \mathcal{J}(w_a, w_b) : \mathcal{P}(a) \times \mathcal{P}(b) \rightarrow T\mathcal{P}(a) \times T\mathcal{P}(b)$$

is of the form

$$X(x, y) = (w_a(x), w_b(x, x, y)) \quad \text{for all } (x, y) \in \mathcal{P}(a) \times \mathcal{P}(b).$$

If $G = \begin{array}{c} \text{---} \text{---} \\ a \quad b \quad c \\ \text{---} \text{---} \end{array}$ and $\mathcal{P} : G_0 \rightarrow \text{Man}$ is a phase space function, then

$$(\mathcal{J}(w_a, w_b, w_c))(x, y, z) = (w_a(x), w_b(x, x, y), w_c(y, z))$$

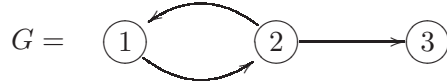
for all $(w_a, w_b, w_c) \in \mathbb{S}(G, \mathcal{P})$ and all $(x, y, z) \in \mathcal{P}(a) \times \mathcal{P}(b) \times \mathcal{P}(c)$.

3 Modularity

3.1 Symmetry groupoid of a network

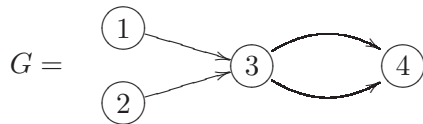
In this section we give one possible version of what it means for some of the open subsystems in the tuple of the constituent subsystems $(w_a)_{a \in G_0} \in \mathbb{S}(G, \mathcal{P})$ on a network (G, \mathcal{P}) to be “the same.” We start with a pair of examples.

3.1.1 Example. Consider the graph



from subsection 1.4. Choose a phase space function $\mathcal{P} : G_0 = \{1, 2, 3\} \rightarrow \text{Man}$ with $\mathcal{P}(1) = \mathcal{P}(2) = \mathcal{P}(3) = M$ for some manifold M . Then a typical tuple of open subsystems in $\mathbb{S}(G, \mathcal{P})$ that defines the dynamics on the network is a triple of the form $(f_1 : M \times M \rightarrow TM, f_2 : M \times M \rightarrow TM, f_3 : M \times M \rightarrow TM)$. It make sense to require that $f_1 = f_2 = f_3$. We can do it because the input trees $I(1), I(2), I(3)$ and the corresponding networks $(I(i), \mathcal{P} \circ \xi_i)$, $1 \leq i \leq 3$, are all isomorphic (here, as before $\xi_i : I(i) \rightarrow G$ are the canonical maps, q.v. Remark 2.3.3).

3.1.2 Example. Consider the graph



Again define a phase function \mathcal{P} by setting $\mathcal{P}(i)$ to be the same manifold M for all i . An element of $\mathbb{S}(G, \mathcal{P})$ is then of the form

$$(f_1 : M \rightarrow TM, f_2 : M \rightarrow TM, f_3 : M \times M \times M \rightarrow TM, f_4 : M \times M \times M \rightarrow TM).$$

Now it does not make sense to require that $f_3 = f_1$ but it does make sense to require that $f_1 = f_2$ and $f_3 = f_4$ (!). Note that in this example the networks $(I(1), \mathcal{P} \circ \xi_1)$ and $(I(2), \mathcal{P} \circ \xi_2)$ are isomorphic as are the networks $(I(3), \mathcal{P} \circ \xi_3)$ and $(I(4), \mathcal{P} \circ \xi_4)$.

If we were to set $\mathcal{P}(1) = \mathcal{P}(2) = \mathcal{P}(3) = M$ and $\mathcal{P}(4) = N \neq M$, then an element of $\mathbb{S}(G, \mathcal{P})$ would be of the form

$$(f_1 : M \rightarrow TM, f_2 : M \rightarrow TM, f_3 : M \times M \times M \rightarrow TM, f_4 : M \times M \times N \rightarrow TN).$$

In this case setting $f_1 = f_2$ would make sense but setting $f_3 = f_4$ would not. And while $(I(1), \mathcal{P} \circ \xi_1)$ and $(I(2), \mathcal{P} \circ \xi_2)$ would still be isomorphic, the networks $(I(3), \mathcal{P} \circ \xi_3)$ and $(I(4), \mathcal{P} \circ \xi_4)$ would not.

3.1.3 Remark. In Example 3.1.1 there are 3^2 isomorphisms

$$\varphi_{ij} : (I(j), \mathcal{P} \circ \xi_j) \rightarrow (I(i), \mathcal{P} \circ \xi_i), \quad 1 \leq i, j \leq 3$$

with

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$$

for all i, j, k , and with

$$\varphi_{ii} = \text{id}$$

for all i (and consequently $\varphi_{ji} = \varphi_{ij}^{-1}$). These 9 maps are an example of a groupoid.

We recall the shortest definition of a groupoid:

3.1.4 Definition. A groupoid is a category with every morphism an isomorphism.

3.1.5 Remark. One may think of a groupoid \mathbb{H} as a directed graph $\{\mathbb{H}_1 \rightrightarrows H_0\}$ together with an associative multiplication of pairs of edges with matched source and target:

$$(a \xleftarrow{\alpha} b \xleftarrow{\beta} c) \mapsto (a \xleftarrow{\alpha\beta} b)$$

an inversion map

$$(a \xleftarrow{\alpha} b) \mapsto (a \xrightarrow{\alpha^{-1}} b)$$

and a unit edge $\text{id}_a : a \rightarrow a$ for every vertex a of \mathbb{H} . We refer to the elements of \mathbb{H}_0 as the *objects* of the groupoid \mathbb{H} and to the elements of \mathbb{H}_1 as *isomorphisms* of \mathbb{H} .

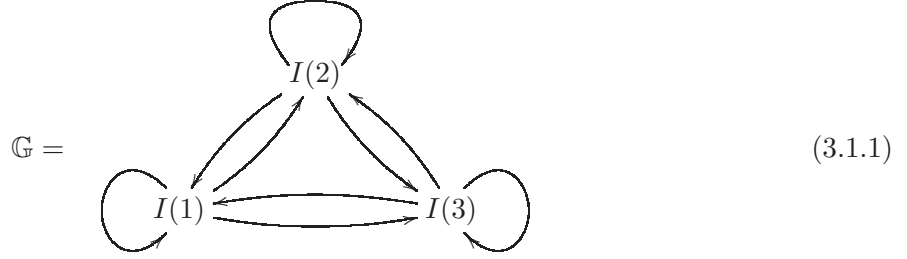
3.1.6 Example. In Remark 3.1.3 the groupoid \mathbb{G} associated to the network (G, \mathcal{P}) has three objects, namely the networks $(I(i), \mathcal{P} \circ \xi_i)$, $1 \leq i \leq 3$, and 9 isomorphisms φ_{ij} , $1 \leq i, j \leq 3$. For the corresponding graph see (3.1.1) below.

3.1.7 Definition (Symmetry groupoid $\mathbb{G}(G, \mathcal{P})$ of a network). The symmetry groupoid $\mathbb{G} = \mathbb{G}(G, \mathcal{P})$ of a network (G, \mathcal{P}) is a category with the following sets of objects and isomorphisms, respectively. The set of objects \mathbb{G}_0 of \mathbb{G} is the set of input networks

$$\{(I(a), \mathcal{P} \circ \xi_a)\}_{a \in G_0}.$$

The set of isomorphism \mathbb{G}_1 of \mathbb{G} is the set of all possible isomorphisms of the input networks.

3.1.8 Example. Consider the network of Example 3.1.1. As we have already pointed out the symmetry groupoid \mathbb{G} of this network has 3 objects and 9 isomorphisms. It can be picture as follows:



3.2 Groupoid-invariant vector fields

Given a network (G, \mathcal{P}) with a groupoid symmetry we should be able to talk about invariant elements of the vector space $\mathbb{S}(G, \mathcal{P})$, the vector space of constituent open subsystems. This is indeed the case. There are several ways of making sense of invariants. The most concrete cuts out the subspace of invariant by appropriate equations. To set this up we need a number of short technical lemmas. We formulate them in a generality that is not needed immediately but will be useful later. The point of the lemmas is to prove that for a given a network (G, \mathcal{P}) there is a natural action of its symmetry groupoid $\mathbb{G}(G, \mathcal{P})$ on the vector bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$.

We start by spelling out what we mean by the action of $\mathbb{G}(G, \mathcal{P})$ on $\text{Control}(G, \mathcal{P})$.

3.2.1 Notation. Denote the category of real vector spaces and linear maps by Vect .

3.2.2 Definition. An action of the groupoid $\mathbb{G}(G, \mathcal{P})$ on the vector bundle $\text{Control}(G, \mathcal{P})$ is a functor

$$\rho : \mathbb{G}(G, \mathcal{P}) \rightarrow \text{Vect},$$

so that

$$\rho(I(a), \mathcal{P} \circ \xi_a) = \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$$

for all nodes a of the graph G . Here as above Vect denotes the category of vector spaces and linear maps.

3.2.3 Remark. The definition amounts to the following:

1. For any two vertices $a, b \in G_0$ and an isomorphism $\varphi : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(b), \mathcal{P} \circ \xi_b)$ in the groupoid \mathbb{G} , there is an isomorphism

$$\rho(\varphi) : \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \rightarrow \text{Ctrl}(\mathbb{P}I(b) \rightarrow \mathbb{P}b);$$

2. If $(I(a), \mathcal{P} \circ \xi_a) \xrightarrow{\varphi} (I(b), \mathcal{P} \circ \xi_b) \xrightarrow{\psi} (I(c), \mathcal{P} \circ \xi_c)$ is a pair of isomorphism in \mathbb{G} then

$$\rho(\psi \circ \varphi) = \rho(\psi) \circ \rho(\varphi);$$

3. If $\varphi : I(a) \rightarrow I(a)$ is the identity isomorphism then $\rho(\varphi)$ is the identity linear map.

The notion of an action of a groupoid on a vector bundle is fairly old. See, for example, [74]. In the case where the vector bundle in question is a collection of vector spaces parameterized by a finite set of objects of the groupoid, as is the case for $\text{Control}(G, \mathcal{P})$, it reduces to Definition 3.2.2 above.

3.2.4 Lemma. Suppose $\psi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is an isomorphism of networks. Then $\mathbb{P}\psi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is a diffeomorphism.

Proof. Let ψ^{-1} denote the inverse of ψ . Then $\psi \circ \psi^{-1} = \text{id}_{(G, \mathcal{P})}$ and $\psi^{-1} \circ \psi = \text{id}_{(G', \mathcal{P}')}$. Hence

$$\text{id}_{\mathbb{P}G} = \mathbb{P}\text{id}_{(G, \mathcal{P})} = \mathbb{P}(\psi \circ \psi^{-1}) = \mathbb{P}(\psi^{-1}) \circ \mathbb{P}\psi.$$

By the same argument

$$\text{id}_{\mathbb{P}G'} = \mathbb{P}\psi \circ \mathbb{P}(\psi^{-1}).$$

Hence $\mathbb{P}\psi$ is invertible with the inverse given by $\mathbb{P}(\psi^{-1})$. \square

3.2.5 Remark. Here is a one line category-theoretic proof of Lemma 3.2.4: since \mathbb{P} is a functor, it takes isomorphisms to isomorphisms.

3.2.6 Lemma. Suppose $(G, \mathcal{P}), (G', \mathcal{P}')$ are two networks, $\xi_a : I(a) \rightarrow G$ the input tree of a vertex a of G , $\xi_{a'} : I(a') \rightarrow G'$ the input tree of a vertex a' of G' and $\varphi : I(a) \rightarrow I(a')$ an isomorphism of trees with $\mathcal{P}' \circ \xi_{a'} \circ \varphi = \mathcal{P} \circ \xi_a$. Then the linear map

$$\text{Ctrl}(\varphi) : \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \rightarrow \text{Ctrl}(\mathbb{P}I(a') \rightarrow \mathbb{P}a') \quad (3.2.1)$$

defined by

$$\text{Ctrl}(\varphi) : (F : \mathbb{P}I(a) \rightarrow T\mathbb{P}a) \mapsto D(\mathbb{P}\varphi|_{\{a\}})^{-1} \circ F \circ \mathbb{P}\varphi \quad (3.2.2)$$

is an isomorphism. Here $\varphi|_{\{a\}} : \{a\} \rightarrow \{a'\}$ is the restriction of φ to the subgraph $\{a\}$ of G (by Remark 2.3.6 φ has to send a to a'); it is an isomorphism.

Proof. By assumption $\varphi : (I(a), \mathcal{P} \circ \xi) \rightarrow (I(a'), \mathcal{P}' \circ \xi')$ is an isomorphism of networks. By Lemma 3.2.4, the maps $\mathbb{P}\varphi$ and $\mathbb{P}\varphi|_{\{a\}}$ are diffeomorphisms. Therefore $\text{Ctrl}(\varphi)$ has an inverse given by

$$(F' : \mathbb{P}I(a') \rightarrow T\mathbb{P}a') \mapsto D(\mathbb{P}(\varphi|_{\{a\}})) \circ F' \circ \mathbb{P}\varphi^{-1}.$$

\square

It follows that we may define the functor $\rho : \mathbb{G}(G, \mathcal{P}) \rightarrow \text{Vect}$ on isomorphisms of the groupoid $\mathbb{G}(G, \mathcal{P})$ by setting

$$\rho(\varphi) := \text{Ctrl}(\varphi).$$

3.2.7 Lemma. Given three networks $(G, \mathcal{P}), (G', \mathcal{P}')$ and (G'', \mathcal{P}'') , and a pair of isomorphism of input networks

$$(I(a), \mathcal{P} \circ \xi) \xrightarrow{\varphi} (I(a'), \mathcal{P}' \circ \xi') \xrightarrow{\psi} (I(a''), \mathcal{P}'' \circ \xi'')$$

we have

$$\text{Ctrl}(\psi \circ \varphi) = \text{Ctrl}\psi \circ \text{Ctrl}\varphi. \quad (3.2.3)$$

Proof. For $F \in \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$ we have

$$\begin{aligned} \text{Ctrl}(\psi \circ \varphi)F &= D(\mathbb{P}(\psi \circ \varphi)|_{\{a\}}^{-1}) \circ F \circ \mathbb{P}(\psi \circ \varphi) \\ &= D((\mathbb{P}\varphi|_{\{a\}} \circ \mathbb{P}\psi|_{\{a'\}})^{-1}) \circ F \circ \mathbb{P}\varphi \circ \mathbb{P}\psi \quad (\text{since } \mathbb{P} \text{ is a contravariant functor}) \\ &= D(\mathbb{P}\varphi|_{\{a\}})^{-1} \circ (D(\mathbb{P}\psi|_{\{a'\}})^{-1}) \circ F \circ \mathbb{P}\varphi \\ &= \text{Ctrl}(\psi)(\text{Ctrl}(\varphi)F). \end{aligned}$$

\square

We are now in the position to prove the main result of the section.

3.2.8 Proposition. *The symmetry groupoid $\mathbb{G}(G, \mathcal{P})$ of a network (G, \mathcal{P}) acts on the vector bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$. The action is given by*

$$\left((I(a), \mathcal{P} \circ \xi_a) \xrightarrow{\varphi} (I(b), \mathcal{P} \circ \xi_b) \right) \mapsto \left(\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \xrightarrow{\text{Ctrl}(\varphi)} \text{Ctrl}(\mathbb{P}I(b) \rightarrow \mathbb{P}b) \right),$$

where $\text{Ctrl}(\varphi)$ is defined by (3.2.2).

Proof. We need to check that the three conditions listed in Remark 3.2.3 hold for $\rho(\varphi) = \text{Ctrl}(\varphi)$. The first one holds by Lemma 3.2.6. The second by Lemma 3.2.7. Note finally that by construction if $\varphi : I(a) \rightarrow I(a)$ is the identity isomorphism then $\text{Ctrl}(\varphi)$ is the identity linear map. We conclude that the functor

$$\rho = \text{Ctrl} : \mathbb{G}(G, \mathcal{P}) \rightarrow \text{Vect}$$

defines an action of the groupoid $\mathbb{G}(G, \mathcal{P})$ on the vector bundle $\text{Control}(G, \mathcal{P})$. \square

Our next step is to define the space of invariant sections of the vector bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$ for this action, which is, by definition, the space of invariant virtual vector fields on the network.

3.2.9 Definition (Invariant virtual vector fields on a network). Let (G, \mathcal{P}) be a network. We define the space $\mathbb{V}(G, \mathcal{P})$ of *groupoid-invariant virtual vector fields* on the network to be

$$\begin{aligned} \mathbb{V}G \equiv \mathbb{V}(G, \mathcal{P}) := \\ \{(w_a) \in \mathbb{S}(G, \mathcal{P}) \mid \text{Ctrl}(\sigma)w_a = w_b \text{ for all } \sigma \in \mathbb{G}(G, \mathcal{P}) \text{ with } \sigma : I(a) \rightarrow I(b)\}. \end{aligned} \quad (3.2.4)$$

3.2.10 Example. Consider the network of Example 3.1.1. It is easy to see that

$$\mathbb{V}(G, \mathcal{P}) = \{(f_1, f_2, f_3) \in \mathbb{S}(G, \mathcal{P}) \mid f_1 = f_2 = f_3\},$$

where, as before $f_i : M \times M \rightarrow TM$ are control systems. Note that in this case the space of invariant virtual vector fields is naturally isomorphic to the space $\text{Ctrl}(\mathbb{P}I(1) \rightarrow \mathbb{P}1) = \text{Ctrl}(M \times M \rightarrow TM)$. Note also that $\text{Ctrl}(M \times M \rightarrow TM)$ is the space of invariant virtual vector field for the network (G', \mathcal{P}) where

$$G' = \text{graph with one vertex and one edge}$$

is the graph with one vertex and one edge and the function \mathcal{P} assigns the manifold M to the one vertex of G' .

3.2.11 Remark. The reader may wonder in what sense the sections in $\mathbb{V}(G, \mathcal{P})$ are “invariant.” There are several ways to answer this question. We start with the most concrete. Note that the space W^H of H -invariant vectors for a representation $\rho : H \rightarrow GL(W)$ of a group H satisfies

$$W^H = \{w \in W \mid \rho(\sigma)w = w \text{ for all } \sigma \in H\}. \quad (3.2.5)$$

It is easy to see now that (3.2.4) is an analogue of (3.2.5) for groupoids.

More abstractly, we note that the space W^H is the limit of the functor $\rho : \underline{H} \rightarrow \text{Vect}$. Here \underline{H} denotes the category with one object $*$ and the set of morphisms $\text{Hom}(*, *) = H$. Similarly it is not hard to see that $\mathbb{V}(G, \mathcal{P})$ as defined above by equation (3.2.4) together with the evident projections $\mathbb{V}(G, \mathcal{P}) \rightarrow \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$ is the limit of the functor $\text{Ctrl} : \mathbb{G}(G, \mathcal{P}) \rightarrow \text{Vect}$.

3.2.12 Remark. We would like to think of the image $\mathcal{J}(\mathbb{V}G)$ of the space $\mathbb{V}G$ of invariant virtual vector fields under the interconnection map $\mathcal{J} : \mathbb{S}(G, \mathcal{P}) \rightarrow \Gamma T\mathbb{P}(G, \mathcal{P})$ as the space of “groupoid-invariant vector fields” on $\mathbb{P}(G, \mathcal{P})$. Note that this is *not* literally correct since there is no natural action of the groupoid $\mathbb{G}(G, \mathcal{P})$ either on the tangent bundle $T\mathbb{P}(G, \mathcal{P})$ or on the space of its sections.

3.2.13 Remark. As we observed in Remark 2.3.5 the graph underlying the input tree network $(I(a), \mathcal{P} \circ \xi_a)$ of a network (G, \mathcal{P}) is a directed tree of height 1. If $\varphi : T_1 \rightarrow T_2$ is an isomorphism of trees of height 1, then φ necessarily sends the root $\text{rt } T_1$ of the first tree to the root $\text{rt } T_2$ of the second tree. Hence if $\varphi : (T_1, \mathcal{P}_1) \rightarrow (T_2, \mathcal{P}_2)$ is an isomorphism of networks and T_1, T_2 are trees of height 1, it makes sense to define

$$\text{Ctrl}(\varphi) : \text{Ctrl}(\mathbb{P}T_1 \rightarrow \mathbb{P}\text{rt } T_1) \rightarrow \text{Ctrl}(\mathbb{P}T_2 \rightarrow \mathbb{P}\text{rt } T_2)$$

by a slight modification of (3.2.2):

$$\text{Ctrl}(\varphi)^F := D\mathbb{P}(\varphi|_{\text{rt } T_1})^{-1} \circ F \circ \mathbb{P}\varphi. \quad (3.2.6)$$

The proof of Proposition 3.2.8 is then easy to modify to show that that Ctrl is a well-defined functor from the groupoid of height 1 tree networks and their isomorphism to the category \mathbf{Vect} of (not necessarily finite dimensional) real vector spaces and linear maps.

3.3 An alternative notion of modularity

Throughout the paper we take the point of view that a network is a directed graph G together with an assignment of a phase space to each vertex of G , that is, a pair

$$(G, \mathcal{P} : G_0 \rightarrow \text{collection of phase spaces}).$$

Golubitsky, Stewart and their collaborators in their work on coupled cell networks additionally attach colors to edges of graphs. They require that maps of networks preserve the colors. In particular edges of input trees acquire colors from their canonical maps into the defining graphs, and symmetry groupoids consist of color preserving isomorphisms. Thus from the point of view of Golubitsky *et al.* we work with monochromatic graphs. The results of this paper do have their colored analogues. The proofs, *mutatis mutandis* are the same. See [75].

4 Fibrations and invariant virtual vector fields

We proved in Proposition 2.1.12 that a map of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ defines a smooth map $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ between their total phase spaces (going in the opposite direction). The map φ , in general, does not induce a map between spaces of vector fields on the phase spaces $\mathbb{P}G$ and $\mathbb{P}G'$. Nor does it induce a map between the spaces of virtual vector fields $\mathbb{S}(G, \mathcal{P})$ and $\mathbb{S}(G', \mathcal{P}')$, let alone the spaces of groupoid-invariant virtual vector fields $\mathbb{V}G$ and $\mathbb{V}G'$. There is, however, a natural class of maps of networks that does. Following Boldi and Vigna [65] we call them *fibrations*. The notion of a graph fibrations is old. It arose independently at different times in different areas of mathematics under different names. See [76] for a discussion.

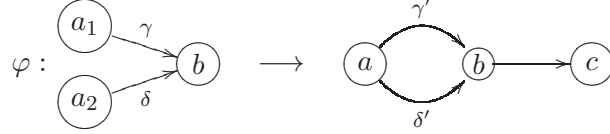
The goal of this section is to prove that a fibration of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ naturally defines a linear map $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$. In the following sections we show that the maps φ^* and $\mathbb{P}\varphi$ and the interconnection maps of the two networks are compatible in the best possible way. Consequently fibrations of networks give rise to maps of dynamical systems.

4.1 Fibrations

4.1.1 Definition. A map $\varphi : G \rightarrow G'$ of directed graphs is a **fibration** if for any vertex a of G and any edge e' of G' ending at $\varphi(a)$ there is a unique edge e of G ending at a with $\varphi(e) = e'$.

A map of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a **fibration** if the corresponding map of graphs $\varphi : G \rightarrow G'$ is a fibration.

4.1.2 Example. The map of graphs



sending the edge γ to γ' and the edge δ to δ' is a graph fibration.

4.1.3 Example. All the maps of graphs in (1.4.4) are graph fibrations. If we define the phase spaces functions on the three graphs by assigning to every node the same manifold M then the corresponding maps of networks are fibrations.

4.1.4. Given any maps $\varphi : G \rightarrow G'$ of graphs and a node a of G there is an induced map of input trees

$$\varphi_a : I(a) \rightarrow I(\varphi(a)).$$

On edges of $I(a)$ the map is defined by

$$\varphi(a, \gamma) := (\varphi(a), \varphi(\gamma))$$

(cf. Definition 2.3.1). Moreover the diagram of graphs

$$\begin{array}{ccc} I(a) & \xrightarrow{\varphi_a} & I(\varphi(a)) \\ \xi_a \downarrow & & \downarrow \xi_{\varphi(a)} \\ G & \xrightarrow{\varphi} & G' \end{array}$$

commutes (the map $\xi_a : I(a) \rightarrow G$ from an input tree to the original graph is defined in Remark 2.3.3).

4.1.5 Lemma. If $\varphi : G \rightarrow G'$ is a graph fibration then the induced maps

$$\varphi_a : I(a) \rightarrow I(\varphi(a))$$

of input trees defined above are isomorphisms for all nodes a of G .

Proof. Given an edge $(\varphi(a), \gamma')$ of $I(\varphi(a))$ there is a unique edge γ of G with $\varphi(\gamma) = \gamma'$ and $t(\gamma) = a$. Consequently $\varphi_a(a, \gamma) = (\varphi(a), \gamma')$. It follows that φ_a is bijective on vertices and edges. \square

4.1.6 Corollary. If a map of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a fibration then

$$\varphi_a : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(\varphi(a)), \mathcal{P}' \circ \xi_{\varphi(a)})$$

is an isomorphism of networks.

Proof. Follows immediately from Lemma 4.1.5 above and the definition of an isomorphism of networks. \square

4.2 Maps between spaces of invariant virtual vector fields

The goal of this subsection is to show that fibrations of networks send groupoid-invariant virtual vector fields to groupoid-invariant virtual vector fields. Namely we prove:

4.2.1 Proposition. *A fibration*

$$\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$$

of networks defines a linear map

$$\varphi^* : \mathbb{S}(G', \mathcal{P}') \rightarrow \mathbb{S}(G, \mathcal{P})$$

between spaces of sections of control bundles, that is, between spaces of virtual vector fields on the networks in question.

Moreover φ^* maps the space $\mathbb{V}(G', \mathcal{P}')$ of groupoid-invariant virtual vector fields to the space $\mathbb{V}(G, \mathcal{P})$.

Proof. Recall (q.v. Definition 2.4.1) that $\mathbb{S}(G, \mathcal{P}) = \prod_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$. We define

$$\varphi^* : \prod_{a' \in G'_0} \text{Ctrl}(\mathbb{P}I(a') \rightarrow \mathbb{P}a') \rightarrow \prod_{a \in G_0} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$$

by

$$(\varphi^* w')_a := \text{Ctrl}(\varphi_a)^{-1}(w'_{\varphi(a)})$$

for all $a \in G_0$. Evidently φ^* is linear.

We now argue that invariant sections get mapped to invariant sections. Consider $w' \in \mathbb{V}(G', \mathcal{P}')$. Let $\sigma : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(b), \mathcal{P} \circ \xi_b)$ be an isomorphism in the groupoid $\mathbb{G}(G, \mathcal{P})$. Since φ is a fibration, the maps

$$\varphi_a : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(\varphi(a)), \mathcal{P}' \circ \xi_{\varphi(a)})$$

and

$$\varphi_b : (I(b), \mathcal{P} \circ \xi_b) \rightarrow (I(\varphi(b)), \mathcal{P}' \circ \xi_{\varphi(b)})$$

are isomorphisms. Therefore

$$\varphi_b \circ \sigma \circ \varphi_a^{-1} : (I(\varphi(a)), \mathcal{P}' \circ \xi_{\varphi(a)}) \rightarrow (I(\varphi(b)), \mathcal{P}' \circ \xi_{\varphi(b)})$$

is an isomorphism of networks, hence an isomorphism in the groupoid $\mathbb{G}(G', \mathcal{P}')$. Since w' is $\mathbb{G}(G', \mathcal{P}')$ invariant by assumption, we have

$$\text{Ctrl}(\varphi_b \circ \sigma \circ \varphi_a^{-1}) w'_{\varphi(a)} = w'_{\varphi(b)}.$$

Since Ctrl is a functor on networks of height 1 trees (cf. Remark 3.2.13, it respects compositions and takes inverses to inverses. Consequently

$$\text{Ctrl}(\varphi_b) \circ \text{Ctrl}(\sigma) \circ \text{Ctrl}(\varphi_a)^{-1} w'_{\varphi(a)} = w'_{\varphi(b)}.$$

Thus

$$\text{Ctrl}(\sigma)(\varphi^* w')_a = \text{Ctrl}(\sigma) \circ \text{Ctrl}(\varphi_a)^{-1} w'_{\varphi(a)} = \text{Ctrl}(\varphi_b)^{-1} w'_{\varphi(b)} = (\varphi^* w')_b,$$

which proves that $\varphi^* w' \in \mathbb{V}(G, \mathcal{P})$. □

4.2.2 Remark. The proof above shows that a fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ also induces a fully-faithful map of groupoids

$$\mathbb{G}(\varphi) : \mathbb{G}(G, \mathcal{P}) \rightarrow \mathbb{G}(G', \mathcal{P}')$$

which is given by

$$\left((I(a), \mathcal{P} \circ \xi_a) \xrightarrow{\sigma} (I(b), \mathcal{P} \circ \xi_b) \right) \mapsto \left((I(\varphi(a)), \mathcal{P} \circ \xi_{\varphi(a)}) \xrightarrow{\varphi_b \circ \sigma \circ \varphi_a^{-1}} (I(\varphi(b)), \mathcal{P} \circ \xi_{\varphi(b)}) \right).$$

4.2.3 Remark. Here is an alternative, more geometric, way to think of Proposition 4.2.1 and its proof. The collection of maps

$$\{\text{Ctrl}(\varphi_a) : \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a) \rightarrow \text{Ctrl}(\mathbb{P}I(\varphi(a)) \rightarrow \mathbb{P}\varphi(a))\}$$

define a map of vector bundles

$$\tilde{\varphi} : \text{Control}(G, \mathcal{P}) \rightarrow \text{Control}(G', \mathcal{P}'),$$

which restricts to an isomorphism on each fiber. Hence $\text{Control}(G, \mathcal{P}) \rightarrow G_0$ is the pullback of $\text{Control}(G', \mathcal{P}') \rightarrow G'_0$. Consequently sections of $\text{Control}(G', \mathcal{P}') \rightarrow G'_0$ pull back to sections of $\text{Control}(G, \mathcal{P}) \rightarrow G_0$. Moreover the vector bundle map $\tilde{\varphi}$ intertwines the actions of the groupoids $\mathbb{G}(G, \mathcal{P})$ and $\mathbb{G}(G', \mathcal{P}')$. Hence invariant sections pull back to invariant sections.

4.2.4 Remark. In section 6 below we show that somewhat surprisingly the map $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ of Proposition 4.2.1 is always surjective. We also characterize the kernel of φ^* . In particular if $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a quotient map (in the setting of coupled cell networks the fibers of such φ are equivalence classes of a balanced equivalence relation) then $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ is an isomorphism.

4.3 Fibrations and maps of dynamical systems

The goal of this subsection is to prove that fibrations of networks give rise to maps between dynamical systems. This is arguably the main result of the paper. Here is a precise statement:

4.3.1 Theorem. *Let $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ be a fibration of networks. Then for any groupoid-invariant virtual vector field $w' \in \mathbb{V}G'$ the map $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ intertwines the vector fields $\mathcal{J}'(w')$ and $\mathcal{J}(\varphi^*w')$:*

$$D(\mathbb{P}\varphi) \circ \mathcal{J}'(w') = \mathcal{J}(\varphi^*w') \circ \mathbb{P}\varphi. \quad (4.3.1)$$

Equivalently the diagram

$$\begin{array}{ccc} T\mathbb{P}G' & \xrightarrow{D\mathbb{P}\varphi} & T\mathbb{P}G \\ \mathcal{J}'(w') \uparrow & & \uparrow \mathcal{J}(\varphi^*w') \\ \mathbb{P}G' & \xrightarrow{\mathbb{P}\varphi} & \mathbb{P}G \end{array} \quad (4.3.2)$$

commutes.

4.3.2 Remark. Note that by Proposition 4.2.1 since w' is groupoid invariant virtual vector field on the network (G', \mathcal{P}') , the pullback φ^*w' is a *groupoid-invariant* virtual vector field on the network (G, \mathcal{P}) , i.e., $\varphi^*w' \in \mathbb{V}(G, \mathcal{P})$.

Proof of Theorem 4.3.1. Recall that the manifold $\mathbb{P}G$ is the product $\prod_{a \in G_0} \mathbb{P}a$. Hence the tangent bundle $T\mathbb{P}G$ is the product $\prod_{a \in G_0} T\mathbb{P}a$. In particular for each node a of the graph G , the canonical projection

$$T\mathbb{P}G \rightarrow T\mathbb{P}a$$

is the differential of the map $\mathbb{P}\iota_a : \mathbb{P}G \rightarrow \mathbb{P}a$. Here, as before, $\iota_a : \{a\} \hookrightarrow G$ is the canonical inclusion of graphs. By the universal property of products, two maps into $T\mathbb{P}G$ are equal if and only if all their components are equal. Therefore, in order to prove that (4.3.2) commutes it is enough to show that

$$D\mathbb{P}\iota_a \circ \mathcal{J}(\varphi^* w') \circ \mathbb{P}\varphi = D\mathbb{P}\iota_a \circ D\mathbb{P}\varphi \circ \mathcal{J}'(w')$$

for all nodes $a \in G_0$. By definition of the restriction $\varphi|_{\{a\}}$ of $\varphi : G \rightarrow G'$ to $\{a\} \hookrightarrow G$, the diagram

$$\begin{array}{ccc} \{a\} & \xrightarrow{\varphi|_{\{a\}}} & \{\varphi(a)\} \\ \downarrow \iota_a & & \downarrow \iota_{\varphi(a)} \\ G & \xrightarrow{\varphi} & G' \end{array} \quad (4.3.3)$$

commutes. By the definition of the pullback map φ^* and the interconnection maps $\mathcal{J}, \mathcal{J}'$ the diagram

$$\begin{array}{ccccc} & T\mathbb{P}a & \xleftarrow{D\mathbb{P}\varphi|_{\{a\}}} & T\mathbb{P}\varphi(a) & \\ & \uparrow (\varphi^* w')_a & & \uparrow w'_{\varphi(a)} & \\ \mathcal{J}(\varphi^* w')_a & \mathbb{P}I(a) & \xleftarrow{\mathbb{P}\varphi_a} & \mathbb{P}I(\varphi(a)) & \mathcal{J}'(w')_{\varphi(a)} \\ & \uparrow \mathbb{P}\xi_a & & \uparrow \mathbb{P}\xi_{\varphi(a)} & \\ \mathbb{P}G & \xleftarrow{\mathbb{P}\varphi} & \mathbb{P}G' & & \end{array} \quad (4.3.4)$$

commutes as well. We now compute:

$$\begin{aligned} D\mathbb{P}\iota_a \circ \mathcal{J}(\varphi^* w') \circ \mathbb{P}\varphi &= (\mathcal{J}(\varphi^* w'))_a \circ \mathbb{P}\varphi && \text{by definition of } \mathcal{J}(\varphi^* w')_a \\ &= D\mathbb{P}(\varphi|_{\{a\}}) \circ \mathcal{J}'(w')_{\varphi(a)} && \text{by (4.3.4)} \\ &= D\mathbb{P}(\varphi|_{\{a\}}) \circ D\mathbb{P}\iota_{\varphi(a)} \circ \mathcal{J}'(w')_{\varphi(a)} && \text{by definition of } \mathcal{J}'(w')_{\varphi(a)} \\ &= D\mathbb{P}(\iota_{\varphi(a)} \circ \varphi|_{\{a\}}) \circ \mathcal{J}'(w')_{\varphi(a)} && \text{since } \mathbb{P} \text{ is a contravariant functor} \\ &= D\mathbb{P}(\varphi \circ \iota_a) \circ \mathcal{J}'(w')_{\varphi(a)} && \text{by (4.3.3)} \\ &= D\mathbb{P}(\iota_a) \circ D\mathbb{P}\varphi \circ \mathcal{J}'(w'). \end{aligned}$$

□

4.3.3 Remark. In Lemma 5.1.1 below we show that surjective fibrations of networks give rise to embeddings of dynamical systems. Since balanced equivalence relations of the groupoid formalism of Golubitsky *et al.* [39–64] define quotient networks, each balanced equivalence relation give rise to a surjective maps of graphs and hence to surjective fibration of networks in our sense. Thus a special case of Theorem 4.3.1 generalizes one direction of the groupoid formalism correspondence between invariant subspaces and balanced equivalence relations from ordinary differential equations to vector fields on manifolds. More specifically Theorem 4.3.1 is a generalization, to manifolds, of

Theorem 5.2 (direction (b)) of [61] and of Theorem 9.2 of [57]. We do not attempt to establish the converse. More specifically, we do not attempt to characterize submanifolds of total phase spaces of networks that are preserved by all groupoid invariant vector fields—we are only speaking to the “forward” direction.

5 Dynamical consequence of Theorem 4.3.1

In this section, we will discuss the implications of Theorem 4.3.1. Consider a fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ of networks. Then φ defines a map $\varphi_0 : G_0 \rightarrow G'_0$ from the set of vertices of the graph G to the set of vertices of the graph G' . In general φ_0 is neither injective nor surjective. However if a graph fibration $\varphi : G \rightarrow G'$ is surjective on vertices, it is automatically surjective on edges. Similarly if a graph fibration $\varphi : G \rightarrow G'$ is injective on vertices, then it is injective on edges as well. From now on we simply talk about injective and surjective graph fibrations.

Next observe that a given fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ can always be factored as a map onto its image followed by the inclusion of the image:

$$(G, \mathcal{P}) \xrightarrow{\varphi} (\varphi(G), \mathcal{P}') \xrightarrow{i} (G', \mathcal{P}')$$

Hence any fibration can be factored as a surjection followed by an injection. We next analyze surjective and injective fibrations of networks.

5.1 Surjective fibrations

5.1.1 Lemma. *Suppose $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a surjective fibration. Then $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is an embedding whose image is a “polydiagonal”*

$$\Delta_\varphi = \{x \in \mathbb{P}G \mid x_a = x_b \text{ whenever } \varphi(a) = \varphi(b)\}.$$

Proof. Assume first for simplicity that G' has only one vertex $*$ and $\mathcal{P}'(*) = M$. Then for any vertex a of G we have

$$\mathcal{P}(a) = \mathcal{P}'(\varphi(a)) = \mathcal{P}'(*) = M,$$

$\mathbb{P}G' = M$ and $\mathbb{P}G = M^{G_0}$, where as before G_0 is the set of vertices of the graph G . By Proposition 2.1.12 the map $\mathbb{P}\varphi : M \rightarrow M^{G_0}$ is of the form

$$\mathbb{P}\varphi(x) = (x, \dots, x)$$

for all $x \in M$.

In general

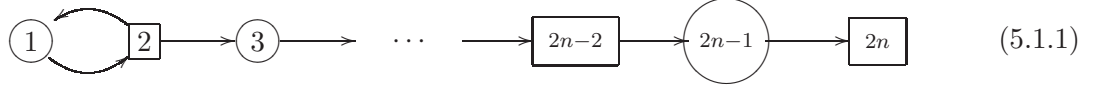
$$\mathbb{P}\varphi : \mathbb{P}G' = \prod_{a' \in G'_0} \mathcal{P}'(a') \rightarrow \prod_{a' \in G'_0} \left(\prod_{a \in \varphi^{-1}(a')} \mathcal{P}'(a') \right) = \mathbb{P}G$$

is the product of maps of the form

$$\mathcal{P}'(a') \rightarrow \prod_{a \in \varphi^{-1}(a')} \mathcal{P}'(a'), \quad x \mapsto (x, \dots, x).$$

□

5.1.2 Example. We consider the following surjective fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ of networks. We take G to be the graph



with $2n$ vertices ($n \geq 2$). We choose a phase space function \mathcal{P} that assigns a manifold M to all odd numbered vertices and a (different) manifold N to all even numbered vertices. We take G' to be the graph



with two vertices and two arrows. We set $\mathcal{P}'(a) = M$ and $\mathcal{P}'(b) = N$. We define the surjective fibration $\varphi : G \rightarrow G'$ by setting

$$\varphi(n) = \begin{cases} a, & n \text{ odd}, \\ b, & n \text{ even} \end{cases}$$

The corresponding total phase space map $\mathbb{P}\varphi : M \times N \rightarrow (M \times N)^n$ is given by the formula

$$\mathbb{P}(x, y) = (x, y, x, y, \dots, x, y).$$

The groupoid $\mathbb{G}(G', \mathcal{P}')$ is trivial. Consequently $\mathbb{V}(G', \mathcal{P}')$ consists of a pair of control systems $w'_a : M \times N \rightarrow TM$ and $w'_b : N \times M \rightarrow TN$. They interconnect to define a vector field $\mathcal{J}'(w') : M \times N \rightarrow TM \times TN$ with

$$\mathcal{J}'(w')(x, y) = (w'_a(x, y), w'_b(y, x))$$

for all $(x, y) \in M \times N = \mathbb{P}G'$.

The groupoid $\mathbb{G}(G, \mathcal{P})$ is *not* trivial: all input networks corresponding to odd numbered vertices are uniquely isomorphic. That is, given the vertices $2k + 1$ and $2\ell + 1$, $k \neq \ell$, there is exactly one isomorphism $\psi_{k\ell} : I(2k + 1) \rightarrow I(2\ell + 1)$ as well $\psi_{\ell k} = \psi_{k\ell}^{-1} : I(2\ell + 1) \rightarrow I(2k + 1)$. Analogous statement holds for input networks corresponding to even numbered vertices.

The pullback map

$$\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$$

is easily seen to be given by

$$\varphi^*(w'_a, w'_b) = (w'_a, w'_b, \dots, w'_a, w'_b),$$

and $\mathcal{J}(\varphi^*w') \in \Gamma T(M \times N)^n$ is given by

$$\mathcal{J}(\varphi^*w')(x_1, y_1, \dots, x_n, y_n) = (w'_a(x_1, y_1), w'_b(y_1, x_1), \dots, w'_a(x_n, y_n), w'_b(y_n, x_n)).$$

It is clear that $\mathbb{P}\varphi(M \times N)$ is an invariant submanifold of the vector field $\mathcal{J}(\varphi^*w')$, as should be expected in light of Theorem 4.3.1.

5.2 Injective fibrations

Consider an injective fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ of networks. Lemma 5.2.1 below shows that the map $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ of total phase spaces is a surjective submersion. Combining this with Theorem 4.3.1 we see that for any groupoid-invariant virtual vector field $w' \in \mathbb{V}(G', \mathcal{P}')$ the map

$$\mathbb{P}\varphi : (\mathbb{P}G', \mathcal{J}'(w')) \rightarrow (\mathbb{P}G, \mathcal{J}(\varphi^*w'))$$

is a projection of dynamical systems. In particular for any singular point x of the vector field $\mathcal{J}(\varphi^*w')$, i.e., the point where the vector field is zero, the fiber $\mathbb{P}\varphi^{-1}(x)$ is an invariant submanifold of the vector field $\mathcal{J}(w')$.

Note also that since the map of graphs $\varphi : G \rightarrow G'$ is injective, $\varphi : G \rightarrow \varphi(G)$ is an isomorphism. Since φ is also a graph fibration, there are no edges of G' with the source in $G' \setminus \varphi(G)$ and target in the image $\varphi(G)$. Thus the image of φ is a subsystem of G' that drives the dynamical system on G' . In other words the notion of an injective fibration makes precise the intuitive idea of a subsystem driving a larger network.

5.2.1 Lemma. *Suppose $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is an injective fibration. Then $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is a surjective submersion.*

Proof. Since $\varphi : G \rightarrow G'$ is injective, the set of nodes G'_0 of G' can be partitioned as the disjoint union of the image $\varphi(G_0)$, which is a copy of G_0 , and the complement. Hence

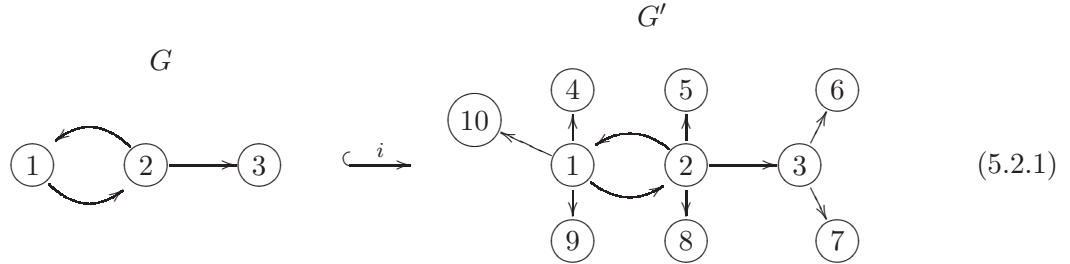
$$\mathbb{P}G' \simeq \prod_{a \in G_0} \mathcal{P}(\varphi(a)) \times \prod_{a' \notin \varphi(G_0)} \mathcal{P}'(a') \simeq \mathbb{P}G \times \prod_{a' \notin \varphi(G_0)} \mathcal{P}'(a').$$

With respect to this identification of $\mathbb{P}G'$ with $\mathbb{P}G \times \prod_{a' \notin \varphi(G_0)} \mathcal{P}'(a')$ the map $\mathbb{P}\varphi : \mathbb{P}G' \rightarrow \mathbb{P}G$ is the projection

$$\mathbb{P}G \times \prod_{a' \notin \varphi(G_0)} \mathcal{P}'(a') \rightarrow \mathbb{P}G.$$

which is a surjective submersion. □

5.2.2 Example. Consider the injective graph fibration



Choose phase space functions $\mathcal{P}, \mathcal{P}'$ so that $i : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a map of networks. By the discussion above, for any choice of a groupoid invariant virtual vector field $w' \in \mathbb{V}(G', \mathcal{P}')$ the dynamics in the subsystem $(\mathbb{P}G, \mathcal{J}(i^*w'))$ drives the entire system $(\mathbb{P}G', \mathcal{J}(w'))$. This is intuitively clear from the graph (5.2.1) since there are no “feedbacks” from vertices 4, ..., 10 back into 1, 2, 3.

5.3 General maps

As we observed in the beginning of the section any fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ can be factored as a surjection onto its image

$$\varphi : (G, \mathcal{P}) \rightarrow (\varphi(G), \mathcal{P}')$$

followed by the inclusion

$$i : (\varphi(G), \mathcal{P}') \hookrightarrow (G', \mathcal{P}').$$

It follows from the two subsections above that for any groupoid invariant virtual vector field $w' \in \mathbb{V}(G', \mathcal{P}')$ the map of dynamical systems

$$\mathbb{P}\varphi : (\mathbb{P}G', \mathcal{J}'(w')) \rightarrow (\mathbb{P}G, \mathcal{J}(\varphi^* w'))$$

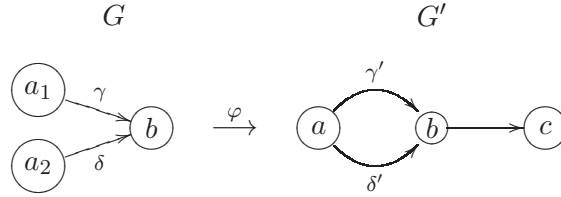
factors as a projection of dynamical systems

$$\mathbb{P}i : (\mathbb{P}G', \mathcal{J}'(w')) \rightarrow (\mathbb{P}\varphi(G), \mathcal{J}'(i^* w'))$$

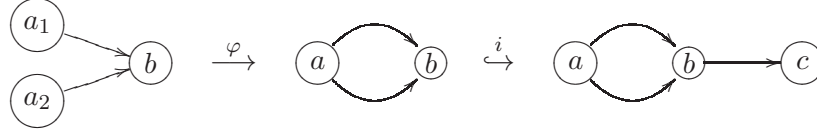
followed by the embedding

$$(\mathbb{P}\varphi(G), \mathcal{J}'(i^* w')) \rightarrow (\mathbb{P}G, \mathcal{J}(\varphi^* w')).$$

5.3.1 Example. Consider the graph fibration



from Example 4.1.2. Choose a phase space function \mathcal{P}' on G' and define $\mathcal{P} : G_0 \rightarrow \text{Man}$ by $\mathcal{P}(a_1) = \mathcal{P}(a_2) = \mathcal{P}'(a)$, $\mathcal{P}(b) = \mathcal{P}'(b)$. Then $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is a fibration of networks. It factors as



6 Spaces of invariant virtual vector fields

The purpose of this section is to characterize further and more precisely the space of groupoid-invariant virtual vector fields on a network (G, \mathcal{P}) and to understand better the pullback maps $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ induced by fibrations of networks $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$.

6.1 The space $\mathbb{V}(G, \mathcal{P})$ as a product of spaces of fixed vectors

It will be useful to introduce a bit more notation.

6.1.1 Notation. Given a network (G, \mathcal{P}) we have an evident bijection between the set G_0 of vertices of the graph G and the set $\mathbb{G}_0 = \{(I(a), \mathcal{P} \circ \xi_a)\}_{a \in G_0}$ of objects of the groupoid $\mathbb{G}(G, \mathcal{P})$. It will be convenient to identify the two sets:

$$\mathbb{G}(G, \mathcal{P})_0 = G_0.$$

6.1.2 Definition (Automorphism group). For a vertex a of a graph G , hence for an object of the symmetry groupoid $\mathbb{G}(G, \mathcal{P})$ of a network (G, \mathcal{P}) we set

$$\text{Aut}(a) := \{\psi : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(a), \mathcal{P} \circ \xi_a) \mid \psi \text{ is an isomorphism of networks} \}$$

Clearly $\text{Aut}(a)$ is a group under composition. We call it the **automorphism group** of the vertex a .

6.1.3 Remark. Note that $\text{Aut}(a)$ is the collection of isomorphisms of the groupoid $\mathbb{G}(G, \mathcal{P})$ with source and target a .

By construction $\text{Aut}(a)$ acts on the vector space $\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)$, the fiber of the bundle $\text{Control}(G, \mathcal{P}) \rightarrow G_0$. We denote the space of fixed vectors by $\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)^{\text{Aut}(a)}$

6.1.4 Remark. In general given an object a of a groupoid $\mathbb{H} = \{\mathbb{H}_1 \rightrightarrows \mathbb{H}_0\}$ we have a group $\text{Aut}(a)$ consisting of isomorphism of \mathbb{H} with source and target a .

6.1.5 Definition. Given a groupoid \mathbb{H} we say that two objects a and b of \mathbb{H} are isomorphic if there is an isomorphism γ of \mathbb{H} with source a and target b .

6.1.6 Remark. It follows easily from the definition of a groupoid that being isomorphic is an equivalence relation on the objects. We denote the collection of isomorphism classes of objects of a groupoid \mathbb{H} by $\mathbb{H}_0/\mathbb{H}_1$ and denote the isomorphism class of an object a by $[a]$.

6.1.7 Lemma. Let (G, \mathcal{P}) be a network. The space $\mathbb{V}(G, \mathcal{P})$ of groupoid invariant virtual vector fields is isomorphic (as a vector space) to the product

$$\bigsqcup_{[a] \in \mathbb{G}_0/\mathbb{G}_1} \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)^{\text{Aut}(a)}.$$

Here as in Remark 6.1.3 $\text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)^{\text{Aut}(a)}$ is the space of vectors fixed by the action of $\text{Aut}(a)$.

Proof. Suppose $w \in \mathbb{V}(G, \mathcal{P})$ is an invariant section of $\text{Control}(G, \mathcal{P}) \rightarrow G_0$. Then for any node a of G_0 and any automorphism $\psi \in \text{Aut}(a)$ we have

$$\text{Ctrl}(\psi)w_a = w_a.$$

Hence $w_a \in \text{Ctrl}(\mathbb{P}I(a) \rightarrow \mathbb{P}a)^{\text{Aut}(a)}$.

If a and b are two isomorphic objects in the groupoid $\mathbb{G}(G, \mathcal{P})$ by way of $\psi : (I(a), \mathcal{P} \circ \xi_a) \rightarrow (I(b), \mathcal{P} \circ \xi_b)$ then

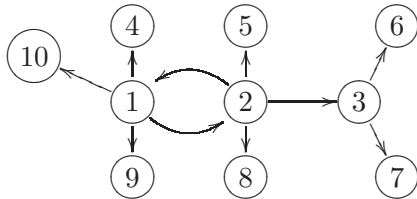
$$w_b = \text{Ctrl}(\psi)w_a.$$

It follows that if we pick representatives $a_1, \dots, a_N \in G_0$ of the equivalence classes in $\mathbb{G}_0/\mathbb{G}_1$ then the restriction map

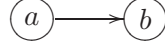
$$\mathbb{V}(G, \mathcal{P}) \rightarrow \bigsqcup_{i=1}^N \text{Ctrl}(\mathbb{P}I(a_i) \rightarrow \mathbb{P}a_i)^{\text{Aut}(a_i)}, \quad w \mapsto (w_{a_1}, \dots, w_{a_N})$$

is an isomorphism of vector spaces. This proves the lemma. \square

6.1.8 Example. Consider the network (G, \mathcal{P}) where G is the graph



and \mathcal{P} assigns the same manifold M to each vertex of G . Then the input trees of G are all of the form



and the corresponding input networks are all isomorphic. Moreover they have trivial automorphism groups. Consequently

$$\mathbb{V}(G, \mathcal{P}) \simeq \text{Ctrl}(M \times M \rightarrow TM).$$

This is quite small compared to the space of all vector fields on the total phase space $\mathbb{P}G \simeq M^{10}$.

6.2 Maps between spaces of invariant virtual vector fields

The goal of this subsection is to understand the pullback map $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ between groupoid-invariant virtual vector fields induced by a fibration $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ of networks. We will see that φ^* is always surjective. To describe its kernel we need the following concept.

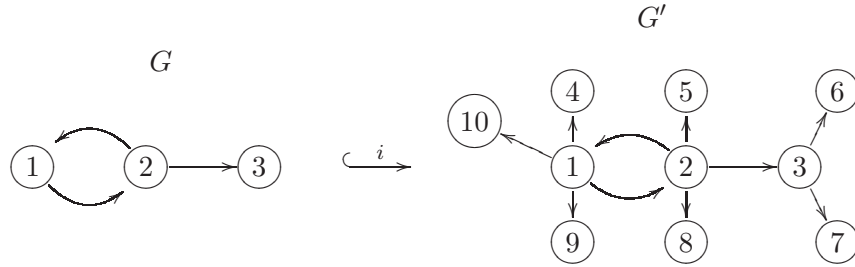
6.2.1 Definition. Let $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ be a fibration of networks. The **essential image** $\text{essim } \varphi \subset G'_0$ of φ consists of all the vertices $a' \in G'_0$ so that there is an isomorphism

$$\psi : (I(a'), \mathcal{P}' \circ \xi_{a'}) \rightarrow (I(\varphi(a)), \mathcal{P} \circ \xi_{\varphi(a)})$$

of input networks for some vertex a of G .

We say that φ is **essentially surjective** if $\text{essim } \varphi = G'_0$.

6.2.2 Example. The map



of networks is not surjective. But it is essentially surjective if $\mathcal{P}'(i) = \mathcal{P}'(j)$ for all $1 \leq i < j \leq 10$, i.e., if we assign the same manifold to all vertices of the graphs.

6.2.3 Theorem. Let $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ be a fibration of networks. Then $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ is surjective. The kernel of φ^* is the space

$$\ker \varphi^* = \{w' \in \mathbb{V}(G', \mathcal{P}') \mid w'_{a'} = 0 \text{ for all } a' \in \text{essim } \varphi\},$$

where $\text{essim } \varphi$ is the essential image of φ defined above.

Proof. For $w' \in \mathbb{V}(G', \mathcal{P}')$ the pullback $\varphi^* w'$ is zero if and only if the component $(\varphi^* w')_a = 0$ for all $a \in G_0$. Since $(\varphi^* w')_a = \text{Ctrl}(\varphi_a)^{-1} w'_{\varphi(a)}$ (q.v. proof of Proposition 4.2.1) and since $\text{Ctrl}(\varphi_a)^{-1}$ is an isomorphism we conclude that

$$\varphi^* w' = 0 \iff w'_{\varphi(a)} = 0 \text{ for all } a \in G_0.$$

Finally note that an invariant section $w' \in \mathbb{V}(G', \mathcal{P}')$ vanishes on the image of φ if and only if it vanishes on the essential image of φ . \square

6.2.4 Corollary. *If $\varphi : (G, \mathcal{P}) \rightarrow (G', \mathcal{P}')$ is an essentially surjective fibration of networks then $\varphi^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ is an isomorphism. In particular φ^* is an isomorphism if φ is surjective.*

6.2.5 Example. Consider the map i of networks in Example 6.2.2. Since i is injective and essentially surjective the map $i^* : \mathbb{V}(G', \mathcal{P}') \rightarrow \mathbb{V}(G, \mathcal{P})$ is an isomorphism. Compare with Example 6.1.8. Clearly the map i is very far from being surjective.

6.2.6 Remark. As we pointed out in Remark 4.3.3 in the groupoid formalism of Golubitsky *et al.* the quotient maps defined by balanced equivalence relations are surjective. Hence the spaces of groupoid invariant vector fields on a network and on its quotient by a balanced equivalence relation are always isomorphic.

Acknowledgments

The authors thank the anonymous referees for many valuable comments that lead to significant improvements in the paper. L.D. was partially supported by NSF grant CMG-0934491.

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